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Exclusion statistics, operator algebras and Fock space representations

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Abstract. We study exclusion statistics within the second quantized approach. We consider operator algebras with positive definite Fock space and restrict them in a such a way that certain state vectors in Fock space are forbidden *ab initio*. We describe three characteristic examples of such exclusion, namely exclusion on the base space which is characterized by states with specific constraint on quantum numbers belonging to base space $\mathcal{M}$ (e.g. the Calogero–Sutherland type of exclusion statistics), exclusion in the single-oscillator Fock space, where some states in single-oscillator Fock space are forbidden (e.g. the Gentile realization of exclusion statistics) and a combination of these two exclusions (e.g. Green's realization of para-Fermi statistics). For these types of exclusions we discuss the extended Haldane statistics parameters g, recently introduced by two of us in 1996 (*Mod. Phys. Lett.* A **11** 3081), and associated counting rules. Within these three types of exclusions in Fock space the original Haldane exclusion statistics cannot be realized.

1. Introduction

Statistics plays a fundamental role in the description of macroscopic or thermodynamic phenomena in quantum many-body systems. It is well known that the stability of matter, built out of protons and electrons, depends crucially on their fermion nature. Also, Bose–Einstein condensation is essentially responsible for the existence of such effects as superfluidity or superconductivity.

Attempts to generalize these conventional (i.e. Bose and Fermi) statistics date back to Gentile's [1] and Green's [2] works on parastatistics in the 1940s and 1950s. Since then a number of papers have been devoted to this topic, culminating in recent years in the discovery of the fractional quantum Hall effect (FQHE) [3], the theory of anyon superconductivity [4] and the Haldane generalization of the Pauli exclusion principle [5].

In principle, there are two distinct approaches to generalized statistics. The starting point of the first approach is some symmetry principle, such as a symmetric group (e.g. parastatistics [2]), braid group (e.g. anyon statistics [4]) or quantum groups (e.g. quon statistics [6]). It can also be characterized by either an operator algebra of creation and annihilation operators with Fock-like representations (second quantization) or monodromy properties of the multiparticle wavefunction (first quantization). For example, Green's parastatistics [2] is based on trilinear commutation relations for particle creation and annihilation operators. For the para-Bose case of order p exactly those representations of the symmetric group S_N with at most p rows

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in the corresponding Young pattern occur, which means that at most p particles can be in an antisymmetric state. In contrast, for the para-Fermi case exactly those representations of the S_N with at most p columns occur, which means that at most p particles can be in a symmetric state. Both cases of parastatistics have some kind of *exclusion* built in by their very definition, i.e. both *a priori* exclude certain irreducible representations (IRREP's) of the symmetric group.

The second approach is based on the state-counting procedure. It is characterized by some Hilbert space of quantum states, generally without a direct connection with creation and annihilation operators acting on Fock-like space. This class includes the recently suggested Haldane generalization of the Pauli exclusion principle, interpolating between Bose and Fermi statistics [5]. The monodromy properties of wavefunctions are not used to define Haldane statistics and the definition of statistics is independent of the dimension of space. Instead, the Haldane statistics of a particle [5] is determined by the *statistics parameter* g, which for the case of one species of particles is defined as

$$g = \frac{d_N - d_{N+\Delta N}}{\Delta N} \tag{1.1}$$

where N is the number of particles and d_N is the dimension of the one-particle Hilbert space obtained by keeping the boundary conditions and quantum numbers of $(N-1)$ particles fixed. For bosons, $g = 0$ and for fermions, the Pauli principle implies $g = 1$.

Alternatively, Wu has defined the exclusion statistics by the interpolating counting formula [7], which gives the number of all independent N-particle states distributed over M quantum states described by M independent oscillators

$$D(M, N, g) = \frac{[M + (N-1)(1-g)!]}{N![M - gN - (1-g)]!}. \tag{1.2}$$

Various aspects of this novel statistics have been investigated [8] and the systems exhibiting it have been described, including one-dimensional (1D) spinons [5] with $g = \frac{1}{2}$, FQHE quasiparticles [9] and anyonic systems [10] (in particular, anyons in the lowest Landau level in a strong magnetic field). In addition, the Haldane concept of statistics applies to the integrable models of Calogero–Sutherland type [11, 12]. There is some evidence that it can also be helpful in understanding the low-temperature physics of 1D Luttinger liquids [13] and the models which exhibits the Mott metal–insulator transition [14]. Although Haldane statistics is defined in an arbitrary number of space dimensions, it is evident that most of these examples are essentially 1D systems. Also, there are still some poorly understood or unanswered questions such as what is the microscopic realization of Haldane statistics or its algebraic and group-theoretical characterization. Furthermore, is there any connection between Green's parastatistics and Haldane statistics, since Green's parastatistics also interpolates between bosons and fermions and generalizes the Pauli exclusion principle.

There have been several attempts to use operator methods to realize Haldane exclusion statistics algebraically [15], but the counting rule $D(M, N, g)$, calculated for these algebras (basically of Gentile type) differs from the Haldane–Wu counting rule. In a previous paper [16] we found that any operator algebra of creation and annihilation operators with a Fock-like representation could be described in terms of *extended* Haldane statistics parameters. Using this result, we described para-Fermi and para-Bose statistics as exclusion statistics of Haldane type and calculated a few extended statistics parameters.

In this paper we continue to study exclusion statistics within the second quantized approach. We consider various operator algebras with positive definite Fock spaces, which have the exclusion principle built into them by their very definition and lead to exclusion statistics. Since there are many ways to perform such exclusion, we describe three characteristic ones, namely exclusion on the base space, the Gentile type of exclusion and a combination of these two exclusions. In section 2 we collect the basic notions of multimode operator algebras [16, 17] which are necessary for sections 3–5, in which we give several examples of exclusions mentioned previously. Inspired by exclusion statistics in the Calogero–Sutherland model [12], in section 3 we define a restricted multimode oscillator algebra of quonic type, which depends on the parameter $\lambda = p/q$ and obeys the generalized exclusion principle. We calculate extended Haldane statistics parameters and discuss the corresponding counting rules for different values of p and q. We recover the Haldane–Wu counting formula for $q = 1$ and integer values of p. We also briefly mention Fermi and Bose-like exclusion algebras. In section 4 we discuss Gentile-type algebras, with the common feature that a restriction is placed on the single-oscillator Fock space. We find that the average value of the extended Haldane statistics parameters and counting rules differs from the original Haldane parameters and counting rules, in agreement with the results of Chen *et al* [15]. In section 5 we discuss Green's and Palev's parastatistics as types of exclusion statistics. Finally, in section 6 we briefly summarize the main results of the paper.

2. Definition of the multimode oscillator algebras

2.1. Fock space

In this section we briefly review the definition of general multimode oscillator algebras possessing Fock-like representations and well-defined number operators [16, 17].

We start with Hermitian conjugated pairs of annihilation and creation operators, $\{a_i, a_i^\dagger | i \in \mathcal{M}\}$, defined on some base space $\mathcal{M}$. We build a Fock-like space starting from the unique vacuum state $|0\rangle$, such that $\langle 0 \mid 0\rangle = 1$, $a_i|0\rangle = 0, \forall i \in \mathcal{M}$.

An arbitrary multiparticle state can be described as a linear combination of monomial state vectors $(a_{i_1}^\dagger \ldots a_{i_n}^\dagger|0\rangle)$, and the corresponding Fock space $\mathcal{F}_n$ is given as

$$\mathcal{F}_n = \left\{ \sum_{i_1 \ldots i_n} \lambda_{i_1 \ldots i_n} a_{i_1}^\dagger \ldots a_{i_n}^\dagger |0\rangle \mid \lambda_{i_1 \ldots i_n} \in \mathbb{C} \right\}. \tag{2.1}$$

The annihilation operators a_i act on the space $\mathcal{F}_n$ in such a way that

$$\begin{aligned} &a_i a_j^\dagger|0\rangle = \delta_{ij}|0\rangle \\ &a_i a_{i_1}^\dagger a_{i_2}^\dagger|0\rangle = \delta_{ii_1} a_{i_2}^\dagger|0\rangle + \Phi^i_{i_1 i_2; i_1} \delta_{ii_2} a_{i_1}^\dagger|0\rangle \end{aligned} \tag{2.2}$$

and so on [16,17].

The Fock space (and the corresponding statistics) depends crucially on the structure of the base space $\mathcal{M}$ on which single oscillators are placed. The simplest base space $\mathcal{M}$ is the 1D lattice. If the lattice is finite, we can take $\mathcal{M} = \{1, 2, \ldots, M\}$. For an infinite lattice we can have $\mathcal{M} = \mathbb{N}$ or $\mathbb{Z}$. In the continuum limit, we have $\mathcal{M} \subseteq \mathbb{R}$. Boundary conditions, being periodic or not, may also be important for statistics. Furthermore, one can consider a D-dimensional lattice (finite or infinite) and the corresponding continuum limit, with various boundary conditions. Finally, for the base space one can consider various manifolds or curved spaces with non-trivial topological properties which may also have important consequences for statistics.

2.2. Algebra

We define the *algebra* of creation and annihilation operators as a normally ordered (Wick ordered) expansion $\Gamma_{ij}(a^\dagger, a) \equiv a_i a_j^\dagger$ (no symmetry principle is assumed):

$$\Gamma_{ij} \equiv a_i a_j^\dagger = \delta_{ij} + C^{ij} a_j^\dagger a_i + C^{ij}_{jk,ki} a_j^\dagger a_k^\dagger a_k a_i + C^{ij}_{jk,ik} a_j^\dagger a_k^\dagger a_i a_k + C^{ij}_{kj,ik} a_k^\dagger a_j^\dagger a_i a_k + C^{ij}_{kj,ki} a_k^\dagger a_j^\dagger a_k a_i + \cdots \tag{2.3}$$

where C's are scalar coefficients. Notice that there is no need to define any relation between the creation (e.g. $\Gamma_{ij}(a^\dagger, a^\dagger)$) or annihilation (e.g. $\Gamma_{ij}(a, a)$) operators as they appear implicitly as norm zero vectors in Fock space. (For the treatment of the class of Wick ordered multimode oscillator algebras of the form $a_i a_j^\dagger = \delta_{ij}\mathbf{1} + \sum_{k,l} C^{kl}_{ij} a_l^\dagger a_k$, see [18].)

We also demand that the algebra (2.3) possesses compatible number operators N_i such that $[N_i, a_j^\dagger] = \delta_{ij} a_i^\dagger$ and $[N_i, a_j] = -\delta_{ij} a_i$.

2.3. Matrix of inner products $\mathcal{A}^{(N)}$ and statistics

For an N-particle state $(a_{i_1}^\dagger \ldots a_{i_N}^\dagger |0\rangle)$ with fixed indices $i_1, \ldots, i_N = 1, 2, \ldots, M$, there are $N!/n_1! n_2! \ldots n_M!$ (in principle different) states obtained by permutations $\pi \in S_N$ acting on the state $(a_{i_1}^\dagger \ldots a_{i_N}^\dagger |0\rangle)$. Here, $n_1, n_2, \ldots, n_M$ are eigenvalues of the number operators N_i, satisfying $\sum_{i=1}^M n_i = N$. From these vectors we form a Hermitian matrix $\mathcal{A}^{(N)}(i_1, \ldots, i_N)$ of their scalar products [16, 17]. As we have already stated, the appearance of null-vectors implies corresponding relations between monomials in $a_i^\dagger$ and reduces the number of linearly independent states in $\pi(a_{i_1}^\dagger \ldots a_{i_N}^\dagger |0\rangle)$. The number of linearly independent states is now given by the rank of the matrix $\mathcal{A}^{(N)}$, i.e. $d_{i_1, \cdots i_N} = \mathrm{rank}\mathcal{A}(i_1, \ldots, i_N)$.

The set of $d_{i_1,\ldots,i_N}$ for all possible $i_1, \ldots, i_N = 1, 2, \ldots, M$ and all integers N completely characterizes the statistics and the thermodynamic properties of a *free* system with the corresponding Fock space. (Note that the statistics, i.e. the set $d_{i_1,\ldots,i_N}$ do not uniquely determine the algebra given by equation (2.3).)

Now, we would like to connect the set $d_{i_1,\ldots,i_N}$ with the notion of Haldane generalized exclusion statistics. Following Haldane's idea [5], we define the dimension of the one-particle subspace of Fock space keeping the $(N-1)$ quantum numbers $i_1, \ldots, i_{N-1}$ inside the N-particle states fixed:

$$d^{(1)}_{i_1,\ldots,i_{N-1}} = \sum_{j=1}^M d_{j,i_1,\ldots,i_{N-1}}. \tag{2.4}$$

We point out that $d_{i_1,\ldots,i_N}$ and $d^{(1)}_{i_1,\ldots,i_{N-1}}$ are integers, i.e. no fractional dimension is allowed by definition.

The number of all independent N-particle states distributed over M quantum states described by M independent oscillators $(i = 1, 2 \ldots, M)$ is given by

$$D(M, N; \Gamma) = \sum_{i_1,\ldots,i_N=1}^M d_{i_1,\ldots,i_N}. \tag{2.5}$$

Note that $0 \leqslant D(M, N) \leqslant M^N$ and $D(M, N)$ is always an integer by definition.

The next step is to define the analogue of the Haldane statistics parameter g. Recall that Haldane introduced the statistics parameter g through the change of the single-particle Hilbert space dimension d_n, equation (1.1). In a similar way we define *extended* Haldane statistics parameters [16] $g_{i_1,\ldots,i_{N-1};j_1\ldots,j_k}$ through the change of the available one-particle Fock-subspace dimension $d^{(1)}_{i_1,\ldots,i_{N-1}}$, equation (2.4), i.e.

$$g_{i_1,\ldots,i_{N-1};j_1\ldots,j_k} = \frac{d^{(1)}_{i_1,\ldots,i_{N-1}} - d^{(1)}_{i_1,\ldots,i_{N-1};j_1\ldots,j_k}}{k}. \tag{2.6}$$

Note that equation (2.6) implies that extended Haldane statistics parameters can be any rational numbers. Examples of the calculation of the matrix $\mathcal{A}^{(N)}$ and extended Haldane statistics parameters $g_{i_1,\ldots,i_{N-1};j_1\ldots,j_k}$ for parastatistics are given in [16].

In the following sections, which constitute the core of the paper, we discuss extended Haldane statistics parameters for various types of generalized exclusion statistics.

3. Restricted algebras and projected Fock spaces

As we have seen, one can define extended Haldane parameters for any algebra. To study exclusion statistics within the second quantized approach, we start with an operator algebra and its positive definite Fock space representation. Then we restrict the algebra in a such way that certain state vectors in Fock space are forbidden, while the rest of Fock space remains unchanged. There are many ways to perform such exclusions. Here we do not pretend to give a complete list of all possible exclusions, but we describe and analyse three main classes of exclusion statistics:

(1) Exclusion on the base space $\mathcal{M}$ (e.g., the Calogero–Sutherland type of exclusion statistics) which is characterized by states with a specific constraint on the positions, momenta or other quantum numbers belonging to the base space (lattice) $\mathcal{M}$. Generally, we can write (no summation over repeated indices)

$$a_i a_j^\dagger a_k^\dagger = \Gamma_{ij}\Theta_{jk}a_k^\dagger$$

where Θ_{jk} is 0 or 1, depending on whether the simultaneous appearance of j and k is, respectively, forbidden or allowed. The creation operators of the above Θ-restricted algebra, acting on the vacuum $|0\rangle$, formally reproduces all the states of the initial Fock space (2.1) of the algebra Γ_{ij}, equation (2.3). However, owing to the appearance of Θ_{jk} in the above restricted algebra, monomial states, which do not obey the Θ_{jk}-restriction, will have zero norm and effectively disappear from the Fock space leading to the projected Fock space. This can be easily seen on two particle states (see, e.g., examples 3.4 and 3.5).

(2) Single-oscillator Fock space restrictions or Gentile-type exclusion (e.g., Karabali–Nair algebra [15] and genons [19]) where some states in single-oscillator Fock spaces are forbidden. Let $a^\dagger a = \phi(N)$, $aa^\dagger = \phi(N+1)$, N being the number operator with integer eigenvalues $n \in \mathbb{N}_0$ and $\phi(n) \geqslant 0$ [20].

Then we can restrict the algebra by $a^\dagger a = \phi(N)\theta(N)$, where $\theta(N)$ is 0 or 1, depending on whether the given N-particle (excitation) state is forbidden or allowed. The simplest case [21] is $\theta(n) = 1, n \leqslant p$ and $\theta(n) = 0, n > p$.

(3) Exclusions on the base space $\mathcal{M}$ and the single Fock space simultaneously, i.e. a combination of exclusions of the first and second type (e.g., Green's and Palev's parastatistics [2,22]). Generally,

$$a_i a_j^\dagger a_k^\dagger = \Gamma_{ij}\Theta_{jk}\Phi(N_a)a_k^\dagger$$

where $\Phi(N_a)$ is a function of the number operators N_a of the ath particle. In some cases it is not necessary to project states out of Fock space, since the algebra itself incorporates such exclusions, for example $a_i a_j^\dagger = \Gamma_{ij}\Phi(N)$, where $\Phi(N)$ is a functional of the total number operator such that $\Phi(n) > 0, n \leqslant p$, and $\Phi(p+1) = 0$.

3.1. Calogero–Sutherland type of fractional statistics

Let us start with the dynamical Calogero–Sutherland (C–S) model in 1D, a well known example of exclusion fractional statistics [11, 12]. The Hamiltonian for N particles on a ring of length L is given by

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j} \frac{2\lambda(\lambda-1)}{d^2(x_i - x_j)} \tag{3.1}$$

where $\hbar^2/2m = 1$ and $d(x) = (L/2\pi)\sin(\pi x/L)$ and $\lambda \geqslant 0$. The spectrum of this Hamiltonian is simple and can be expressed in terms of pseudomomenta k_j, $j = 1, 2, \ldots, N$, in the following form

$$E(k_1, \ldots, k_N) = \sum_{j=1}^{N} k_j^2 \tag{3.2}$$

where $k_1 < k_2 < \cdots < k_N$ and $k_{i+1} - k_i = \kappa(\lambda + n_{i+1})$, $n_{i+1} \in \mathbb{N}_0$, $\kappa = 2\pi/L$ and $k_1 = \kappa\,(\lambda(N-1)/2 + n_1)$, implying fractional statistics. The ground-state energy is for $n_2 = n_3 = \cdots = n_N = 0$ and reads

$$E(k_1^0, \ldots, k_N^0) = \sum_{j=1}^{N} (k_j^0)^2 = \frac{\pi^2\lambda^2 N(N^2-1)}{3L^2}.$$

The structure of the spectrum of the Calogero model on the 1D line in the harmonic potential with frequency ω or in a box is similar to (3.2), with κ depending on ω or on the size L of the box. Note that $\lambda = 0$ implies bosons (on the momentum lattice in units of κ) with $k_{i+1} - k_i = \kappa n_{i+1}$ and the ground energy $E_0^{\mathrm{B}} = 0$. The value $\lambda = 1$ implies fermions with $k_{i+1} - k_i = \kappa(1 + n_{i+1})$ and ground energy $E_0^{\mathrm{F}} > 0$. For both free bosons and fermions we can write the corresponding creation and annihilation operators which satisfy Bose and Fermi algebras, respectively.

We are inspired and motivated by the relation $k_{i+1} - k_i = \kappa(\lambda + n_{i+1})$ to construct the algebra of creation and annihilation operators characterized by $\lambda > 0$, $\lambda \in \mathbb{R}^+$. To do this, we start with the quon algebra [6] of creation and annihilation operators a_i, $a_i^\dagger$ on the real line satisfying

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij} \qquad i, j \in \mathbb{R}. \tag{3.3}$$

If $|q| < 1$, the corresponding Fock space is positive definite and for a generic N-particle state with mutually different indices there are $(N!)$ linearly independent states [23].

Without loss of generality, we can restrict ourselves to the choice $q = 0$ [6, 24]. Then the restricted algebra, of type (1) (i.e. exclusion on the base space $\mathcal{M}$), corresponding to the algebra $a_i a_j^\dagger = \delta_{ij}$, becomes

$$a_i a_j^\dagger a_k^\dagger = \delta_{ij}\Theta_{jk} a_k^\dagger \tag{3.4}$$

where

$$\Theta_{jk} = \begin{cases} 1 & \text{if } k - j = \kappa(\lambda + n) \\ 0 & \text{otherwise} \end{cases}$$

with $\kappa > 0, \lambda > 0, n \in \mathbb{N}_0$. The allowed N-particle states in Fock space are of the type $(a_{i_1}^\dagger \ldots a_{i_N}^\dagger|0\rangle)$, $i_1 \ldots i_N \in \mathbb{R}$, with $i_{\alpha+1} - i_\alpha = \kappa(\lambda + n)$, $n \in \mathbb{N}_0$ (all other states are null states and hence forbidden). It is obvious that $d_{i_1\ldots i_N} = 1$. If $\mathcal{M} = \mathbb{R}$ (or an infinite lattice), then $d^{(1)}_{i_1\ldots i_N} = \infty$ and the extended statistics parameters $g_{i_1\ldots i_N; j_1\ldots j_k}$ are not well defined. However, it is possible to define these parameters in the following way. We choose the N-particle state $(a^\dagger_{i_1}\ldots a^\dagger_{i_N}|0\rangle)$ and then take the sufficiently large cut-off from the left and from the right, which includes the given N-particle state. So, we obtain a finite segment or a finite lattice with M sites, $M \gg N$.

We shall discuss several cases, depending on the values of λ. The case when $\lambda = p/q$ ($p, q \in \mathbb{N}$) is *rational* is relatively simple. In this case, Ha [12] suggested normalization of the pseudo-momenta such that the neighbour momenta satisfy $i_{\alpha+1} - i_\alpha = p + n_\alpha q$, $n_\alpha \in \mathbb{N}_0$ and $\mathcal{M}$ reduces to an infinite lattice, $\mathcal{M} = \mathbb{Z}$, with Fermi oscillators placed on each site. For an N-particle state $(a^\dagger_{i_1}\ldots a^\dagger_{i_N}|0\rangle)$ the number of blocked oscillators is

$$Np + (p-1) + \sum_{\alpha=1}^{N-1}\Delta_\alpha \tag{3.5}$$

where

$$\Delta_\alpha = \begin{cases} n_\alpha q & \text{if } p \neq q,\ p \neq 1 \\ n_\alpha(p-1) & \text{if } p = q. \end{cases}$$

In particular, if $p = q = 1$ only N-oscillators are blocked. Note that the case $p = q \neq 1$ does not correspond to the standard Fermi oscillators, but the group of p oscillators behaves like an ordinary Fermi oscillator. If $p = 1$, $q \neq 1$, then N oscillators are blocked but internal oscillators inside the neighbours are strongly correlated.

For the closest N-particle states $n_1 = n_2 = \cdots = n_{N-1} = 0$ and for the finite lattice with M sites $\{1, 2, \ldots, M\}$, we obtain the dimension of the one-particle subspace, equation (2.5), as

$$d_N^{(1)} = \Theta(i_1 - p) + \left[\frac{i_1 - p}{q}\right]^- + \Theta(M - i_1 - Np + 1) + \left[\frac{M - i_1 - Np + 1}{q}\right]^- . \tag{3.6}$$

Then, it is easy to find extended Haldane statistics parameters, equation (2.6)

$$g_{N\to N+j} = d_N^{(1)} - d_{N+j}^{(1)}$$
$$= \begin{cases} \left[\dfrac{i_1 - p}{q}\right]^- - \left[\dfrac{i_1 - 2p}{q}\right]^- \\ \qquad \text{if site '} j \text{ is left'} \\ \left[\dfrac{M - i_1 - Np + 1}{q}\right]^- - \left[\dfrac{M - i_1 - (N+1)p + 1}{q}\right]^- \\ \qquad \text{if site '} j \text{ is right'.} \end{cases} \tag{3.7}$$

Hereafter, $[x]^\pm$ denotes the minimal (+)/maximal (−) integer greater/smaller than a given number x, respectively. We observe that $g_{N\to N+j}$ depends on M and p, q as well, and not just on the ratio $\lambda = p/q$. Moreover, there does not exist a limit value when $M \to \infty$. However, one can define the average value of the statistics parameter $\bar{g}$ for $M, M+1, \ldots, M+q-1$, since g is periodic in M with period q. We assume that j is always 'on the right' and for $M \gg Np$, $M \gg q$ we find

$$\bar{g}_{N\to N+j} = \frac{1}{q}\sum_{\alpha=1}^{q} g_{N\to N+j}(\alpha) = \frac{p}{q} = \lambda. \tag{3.8}$$

This follows from the identity

$$\sum_{i=1}^{q}\left[\frac{M-i}{q}\right]^- - \left[\frac{M-i-p}{q}\right]^- = p \qquad p \in \mathbb{N}_0, q \in \mathbb{N}.$$

Similarly, one can find $g_{i_1,i_2,\ldots,i_N;j}$ for arbitrary N-particle states. They depend on M, p, q and $n_1, n_2, \cdots n_{N-1}$. The average value $\bar{g}$ depends generally on p, q and n_N. Simple consideration [12] gives, for general p, q the Haldane statistics parameter g^{Hald}

$$\begin{aligned} g^{\text{Hald}}_{N\to N+1} &= \lim_{M\to\infty}\left(\frac{M - Np - (p-1) - \sum_{\alpha=1}^{N-1} n_\alpha \cdot 2}{q}\right. \\ &\qquad \left. - \frac{M - (N+1)p - (p-1) - \sum_{\alpha=1}^{N} n_\alpha \cdot 2}{q}\right) \\ &= \frac{p}{q} + n_N \end{aligned} \tag{3.9}$$

which is variable. The above consideration gives, for $p = 1$, $g^{\text{Hald}} = 1/q$ and for $p = q$, $g^{\text{Hald}} = 1$. Hence, only if $p = 1$, $\lambda = 1/q$, does the corresponding C–S model have the Haldane statistics parameter $g^{\text{Hald}} = 1/q$ and $\bar{g} = g^{\text{Hald}}$. Moreover, two statistical models with the same g^{Hald} (for example, $g^{\text{Hald}} = 1$) are not the same. Namely, the counting rule $D(M, N; p, q)$ also depends on both p and q (not only on the ratio $\lambda = p/q$):

$$\begin{aligned} D(M, N; p, q) &= \sum_{i=1}^{M-(N-1)p} \begin{pmatrix} \left[\dfrac{M - i - p(N-1)}{q}\right]^- + N - 1 \\ N-1 \end{pmatrix} \\ &= (M - p(N-1)) \binom{N-1+\alpha}{N-1} - q(N-1)\binom{N-1+\alpha}{N} \end{aligned} \tag{3.10}$$

where $\alpha = [(M - 1 - p(N-1))/2]^-$. The above equation follows from the identity [25]

$$\sum_{i=0}^{\alpha-1} \binom{n+i}{n} = \alpha\binom{n+\alpha}{n} - n\binom{n+\alpha}{n+1} = \binom{n+\alpha}{n+1}$$

with $\alpha \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

If $p = q$,

$$D(M, N; q, q) = M\binom{N-1+\alpha}{N-1} - q(N-1)\binom{N+\alpha}{N} \tag{3.11}$$

with $\alpha = [(M-1)/q]^- - N + 1$.

For $p = q = 1$,

$$D(M, N; 1, 1) = \binom{M}{N}.$$

If $q = 1$, $p \in \mathbb{N}_0$

$$D(M, N; p, 1) = \binom{M + (1-p)(N-1)}{N} \tag{3.12}$$

and only if $q = 1$, does equation (3.10) coincide with an *ad hoc* interpolation formula by Haldane and Wu [5]. The simple interpolation of the above equation is obtained by $\Theta_{jk} = 1$ if $k - j \geqslant p$ and $\Theta_{jk} = 0$, $k - j < p \in \mathbb{N}$, i.e. when a single particle blocks p-units (for fermions, $p = 1$). We point out that the case $p = 0$, $q = 1$ makes sense and reproduces the Bose statistics

$$D(M, N; 0, 1) = \binom{M+N-1}{N}$$

but the case $p = 0$ and $q \neq 1$ corresponds to a generalized Bose statistics since

$$D(M, N; 0, q) = M \begin{pmatrix} N - 1 + \left[\dfrac{M-1}{q}\right]^- \\ N - 1 \end{pmatrix} - q(N-1) \begin{pmatrix} N - 1 + \left[\dfrac{M-1}{q}\right]^- \\ N \end{pmatrix}. \tag{3.13}$$

We note that for fractional values of $\lambda = p/q$, the counting rule, equation (3.10), is completely different, even asymptotically, from the Haldane–Wu formula.

If the coupling constant λ is an *irrational* positive number, then the Ha lattice construction and the counting formula, equation (3.10), cannot be applied. In this case it is more appropriate to define

$$g_{n\to n+1} = \lim_{M\to\infty} \{(M - (n-1)\lambda) - (M - n\lambda)\} = \lambda \tag{3.14}$$

where λ is the occupation width of the one-particle state, and the counting rule

$$D(M, N; \lambda) = \begin{pmatrix} M + N - 1 - [(N-1)\lambda]^+ \\ N - 1 \end{pmatrix}. \tag{3.15}$$

Here we have assumed that the first particle can occupy M-states and that the whole N-particle state is smaller than M. It is interesting to note that if $\lambda \in \mathbb{N}_0$, the last equation coincides with equation (3.12) and with the Haldane–Wu formula. However, if λ is not an integer, then equation (3.15) differs from both (3.12) and the Haldane–Wu formula. Equation (3.14) has the advantage of being well defined for any real $\lambda \geqslant 0$ and if $\lambda = p/q$, then $D(M, N)$ depends only on λ. In this case, one can define an effective parameter $\lambda_{\text{eff}} = [(N-1)\lambda]^+/(N-1)$.

Remark. The case $p = q = 1$ ($p = 0, q = 1$) corresponds to non-standard fermions (bosons) since the operators $a_i^\dagger$ do not satisfy the commutation relations for ordinary fermions (bosons), although the statistical properties are the same as for ordinary fermions (bosons) with $g = 1$ ($g = 0$).

In the following subsections we briefly mention possible generalizations of exclusion fractional statistics by constructing projected Fock spaces.

3.2. Fermi-like exclusion statistics

Let us define a monotonic series (or a finite set)

$$X = \{x_n | 0 < x_1 < x_2 < \cdots < x_n < x_{n+1} \cdots\}.$$

Then, we can easily generalize the condition, equation (3.4), to

$$\Theta_{jk} = \begin{cases} 1 & \text{if } k - j \in X \\ 0 & \text{otherwise.} \end{cases}$$

This restriction leads to the N-particle states

$$a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0\rangle \qquad i_{\alpha+1} - i_\alpha \in X.$$

(The energy dependence on i_α, i.e. the dispersion relation, is not specified.) All other states are null-states. The meaning of these restrictions is that only ordered states survive and that distances between neighbours are 'quantized' according to the rule $i_{\alpha+1} - i_\alpha \in X$. This rule generalizes the Pauli exclusion principle and we call the corresponding statistics X-type restricted Fermi statistics. The statistics parameters g and the counting rules $D(M, N; X, F)$ can be found using the results of [25]. The special case of this statistics is the C–S type of fractional statistics (section 3.1).

3.3. Bose-like exclusion statistics

This is, in principle, the same kind of exclusion as in subsection 3.2, but with the only difference being that in a given state there may be an arbitrarily large number of particles (excitations), i.e. the Θ projector satisfies

$$\Theta_{jk} = \begin{cases} 1 & \text{if } k - j \in X \bigcup \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding allowed states are

$$(a_{i_1}^{\dagger})^{n_1} \ldots (a_{i_N}^{\dagger})^{n_N} |0\rangle \qquad n_1, \ldots, n_N \in \mathbb{N} \qquad i_{\alpha+1} - i_\alpha \in X.$$

All other states are null-states. The above restrictions lead to the X-restricted Bose statistics. The counting rule for N-particle states defined on M-neighbouring sites is

$$D(M, N; X, B) = \sum_{k=1}^{N} \binom{N-1}{k-1} D(M, k; X, F) \tag{3.16}$$

where $D(M, k; X, F)$ is the counting rule for the X-restricted Fermi statistics (subsection 3.2). The factor $\binom{N-1}{k-1}$ follows from the identity after equation (3.10).

Both X-restricted Fermi and Bose statistics are examples of permutation non-invariant statistics. The special case $X = \mathbb{N}$ reproduces Fermi (Bose) statistics but the algebra of creation and annihilation operators differs from the ordinary Fermi (Bose) algebra.

3.4. Restricted Fermi algebra

One can start from the permutation invariant Fermi algebra $a_i a_j^{\dagger} = \delta_{ij} - a_j^{\dagger} a_i$ and restrict it in different ways. For example

$$a_i a_j^{\dagger} a_k^{\dagger} = (\delta_{ij} - a_j^{\dagger} a_i)\Theta_{jk} a_k^{\dagger} \tag{3.17}$$

where

$$\Theta_{jk} = \begin{cases} 1 & \text{if } |k - j| \geqslant p \\ 0 & \text{otherwise.} \end{cases}$$

The creation (annihilation) operators anti-commute as ordinary fermions, whereas the operators satisfying equation (3.4) have no commutation relations at all. The algebra (3.17) is different from the algebra (3.4), but their corresponding statistics are the same. The counting rule is given by equation (3.12).

The opposite example is

$$\Theta_{jk} = \begin{cases} 1 & \text{if } |k - j| \leqslant p \\ 0 & \text{otherwise} \end{cases}$$

which implies that many-particle states satisfy the condition $N \leqslant p + 1$ and all other states are forbidden.

3.5. Restricted Bose algebra

Starting with the Bose algebra $a_i a_j^{\dagger} = \delta_{ij} + a_j^{\dagger} a_i$, we can restrict it to a permutation invariant form

$$a_i a_j^{\dagger} a_k^{\dagger} = (\delta_{ij} + a_j^{\dagger} a_i)\Theta_{jk} a_k^{\dagger} \tag{3.18}$$

where Θ_{jk} is given, for example, as in subsection 3.4.

In the first example ($\Theta_{jk} = 1$, $|k - j| \geqslant p$), the creation (annihilation) operators commute and the corresponding statistics are the same as for the quon-projected construction, equation (3.4). The second example ($\Theta_{jk} = 1$, $|k - j| \leqslant p$) is equivalent with any of the $(p + 1)$-neighbouring Bose oscillators inside the initial lattice.

Generally, one can start with any algebra

$$a_i a_j^\dagger = \Gamma_{ij}(a^\dagger; a)$$

restrict it in the following way,

$$a_i a_j^\dagger a_k^\dagger = \Theta_{jk}\Gamma_{ij}(a^\dagger; a)a_k^\dagger$$

and proceed as in the above examples. We also note that the restricted Fermi and Bose statistics can be defined not only on a line, but also on a circle or a lattice with periodic boundary conditions. A special kind of statistics of this type with

$$\Theta_{jk} = \begin{cases} 1 & \text{if } |k - j| \geqslant p \\ 0 & \text{otherwise} \end{cases}$$

is given in [26].

Remarks. The construction of the C–S-type fractional statistics and generalization proposed in subsections 3.2 and 3.3 (but not 3.4 and 3.5) relies crucially on the oriented 1D space (lattice). However, we point out that this obstacle can be evaded and we suggest some interesting physical speculations.

If the creation (annihilation) operators are defined on $\mathcal{M}_1 \otimes \mathcal{M}_2$, where $\mathcal{M}_1$ is D-dimensional and $\mathcal{M}_2$ is 1D space, then one can apply the construction described in subsections 3.2 and 3.3. For example, there can be ordinary bosons and fermions in the $\mathcal{M}_1$ direction but fractional ones in the $\mathcal{M}_2$ direction:

$$a_{i\alpha}a_{j\beta}^\dagger a_{k\gamma}^\dagger = \delta_{\alpha\beta}(\delta_{ij} \pm a_{j\beta}^\dagger a_{i\alpha})\Theta_{\beta\gamma}a_{k\gamma}^\dagger$$
$$\alpha, \beta, \gamma \in \mathcal{M}_2 \qquad i, j, k \in \mathcal{M}_1$$

where $\Theta_{\beta\gamma}$ is described in subsection 3.4. The whole space is not isotropic, i.e. there is a preferable direction $\mathcal{M}_2$. This is one of the main assumptions for the appearance of generalized statistics.

4. Gentile-type statistics: restrictions on each single oscillator

Gentile suggested the first interpolation between Bose and Fermi statistics. It is characterized by the maximal occupation number m of particles (excitations) in a given quantum box. The maximal number of many-particle states is $N_{\max} = mM$. The m states available by one oscillator can be interpreted as internal degrees of freedom. For a single oscillator, $g_{n\to n+1} = d_n^{(1)} - d_{n+1}^{(1)} = 0$ if $n + 1 < m$ and 1 if $n + 1 = m$. Hence, $g^{\text{Hald}} = \bar{g} = 1/m$.

However, if there are M oscillators, the N-particle state is characterized by $1^{N_1}\ldots M^{N_M}$ such that $\sum_{i=1}^M N_i = N$, $N_i \leqslant m$, where i enumerates oscillators $1, 2, \ldots, M$. Alternatively, we can write $0^{n_0}1^{n_1}\ldots m^{n_m}$, such that $\sum_{\alpha=0}^m n_\alpha = M$, $\sum_{\alpha=0}^m \alpha n_\alpha = N$, where n_α denotes the number of oscillators with α-particles (excitations). Let $N_{i_1} \geqslant N_{i_2} \geqslant \cdots \geqslant N_{i_n}$, then n_α is the number of boxes (oscillators) with α particles, $N_{k+1} = N_{k+2} = \cdots = N_{k+n_\alpha} = \alpha$. Then $d_N^{(1)} = M - n_m$ and

$$g_{N\to N+i} = \Delta n_m = 1 \tag{4.1}$$

if 'i' is added to the $(m-1)$ filling and

$$g_{N\to N+i} = \Delta n_m = 0 \tag{4.2}$$

if 'i' is added to n_α, $\alpha \leqslant m-2$. We find

$$\bar{g}_{N\to N+1} = \frac{n_{m-1}}{M - n_m}. \tag{4.3}$$

Note that the average value is different from $1/m$, even if we perform averaging over different N (except for the case of the single oscillator $M = 1$).

Generally, the Gentile-type statistics can be defined by

$$a_i a_j^\dagger = \Gamma_{ij}(a^\dagger; a)\Theta(N, m) \tag{4.4}$$

where $\Theta(N, m) > 0$ for $N \leqslant m$ and $\Theta(m, m) = 0$. The simplest functions with these properties are step-functions $\Theta(N - m)$ and $\Theta(N, m) = 1 - (N/m)$.

As an example, we consider the restricted Bose oscillator

$$aa^\dagger = (1 + a^\dagger a)\,\Theta(m - N) \tag{4.5}$$

with Fock space spanned by $|0\rangle, a^\dagger|0\rangle, \ldots, (a^\dagger)^m|0\rangle$. This oscillator coincides with the truncated Bose oscillator with cut-off [21]

$$aa^\dagger = (1 + a^\dagger a) - (N+1)\,\delta_{N,m} \tag{4.6}$$

or $aa^\dagger = \Theta(m - N + 1)$. The counting rule is

$$D(N, M, m) = \sum_{\sum n_\alpha = M; \sum \alpha n_\alpha = N} \left(\frac{M!}{n_0! n_1! \ldots n_m!}\right)$$
$$D^F(M, N) \leqslant D(M, N, m) \leqslant D^B(M, N). \tag{4.7}$$

The general properties of the Gentile-type algebra are that: (i) the extended Haldane parameters are not constant; (ii) the average value of the extended Haldane statistics parameters differs from $1/m$, $m \in \mathbb{N}$, except for a single oscillator for which $g_{n\to n+1} = \delta_{n,m}$ and $\bar{g} = 1/m$ for $n \leqslant m$; (iii) the counting rule differs from the Haldane–Wu formula for $m \neq 1$; and (iv) thermodynamic properties are different from the Haldane–Wu thermodynamics [8]. These results are in agreement with the results obtained by Chen *et al* [15].

4.1. Karabali–Nair realization of Gentile statistics

All algebras with the Gentile-type statistics satisfy $(a_i)^m \neq 0$, but $(a_i)^{m+1} = 0$ for every $i = 1, 2, \ldots, M$, and the states $a_i^\dagger a_j^\dagger|0\rangle$ and $a_j^\dagger a_i^\dagger|0\rangle$, $i \neq j$, describe the same physical state. Karabali and Nair constructed a special type of Gentile statistics in one dimension which is also of anyonic type. The corresponding statistics has all the properties of Gentile statistics and differs from the original Haldane statistics.

The simplest algebra with Gentile statistics in one dimension is of the form [17, 27]

$$a_i a_j^\dagger - \mathrm{e}^{\mathrm{i}\lambda sgn(i-j)} a_j^\dagger a_i = 0 \qquad \lambda = \pm\frac{2\pi}{m+1} \qquad m \in \mathbb{N}. \tag{4.8}$$

For this algebra, we have $a_i a_i^\dagger = \Phi(N_i)$, $\Phi(n) = \sin(n\lambda/2)/\sin(\lambda/2) > 0$ for $n < m$ and $\Phi(m+1) = 0$.

Remarks. If the boxes are filled with a small number of particles, after adding a few new particles, the system behaves like a Bose system. In contrast to this, if all boxes are filled with $(m-1)$ particles the system behaves like a Fermi system. For Gentile statistics for a large number of states $g_{N\to N+i} = 0$ and for some N-particle states $g_{N\to N+i} > 0$.

Finally, let us mention that besides the 'local' restrictions described in this paper, there are 'global' restrictions on Fock space. Examples are Green's para-Bose and para-Fermi statistics [2] of order p, which can be realized through projections of complete quon Fock space [26] and parastatistics in which only states with $N \leqslant N_0$ particles are allowed [22], regardless of their local structure.

5. Exclusions on the base space and single-oscillator Fock space

The construction of exclusion statistics performed in the preceding sections can be combined to include restrictions between neighbours, as well as the cut-off of single oscillators.We present two examples of such exclusions, which include parastatistics introduced by Palev [22, 28]. Consider the algebra

$$a_i a_j^\dagger a_k^\dagger = f(N)(\delta_{ij} - a_j^\dagger a_i) a_k^\dagger \tag{5.1}$$

with $f(n) > 0$, $n < p$ and $f(p) = 0$. The simplest choice is the step function $f(N) = \Theta(p - N)$ ($\Theta(x) = 0,\ x \leqslant 0$ and $\Theta(x) = 1,\ x > 0$).

We point out that the corresponding statistics is Fermi statistics restricted up to $N \leqslant p$ N-particle states. Hence, the counting rule is simply

$$D^{\mathrm{F}}(M, N) = \binom{M}{N}, N \leqslant p$$

and $D^{\mathrm{F}}(M, N) = 0$ if $N > p$. The above statistics is characterized by the Haldane statistical parameter $g = 1$

$$g_{n\to n+k} = \frac{d_n - d_{n+k}}{k} = \frac{(M - n + 1) - (M - n - k + 1)}{k} = 1 \tag{5.2}$$

if $n + k \leqslant p$. If $n + k = p + 1$, then $g_{n\to n+k} = (M - n + 1)/(p - n + 1)$, $n = 1, 2, \ldots, p$ is fractional but g is not constant any more. Hence, this is not an example for the original Haldane statistics for which the statistics parameter is g = constant. Moreover, the above statistics is also not the statistics of Karabali–Nair type, where $a_i^p \neq 0$, $a_i^{p+1} = 0$, and for any $N \leqslant Mp$ the N-particle state is allowed, since we already have $a_i^2 = 0$ and $N \leqslant p$.

The second example is the Bose counterpart of the algebra (5.1), namely

$$a_i a_j^\dagger a_k^\dagger = f(N)(\delta_{ij} + a_j^\dagger a_i) a_k^\dagger \tag{5.3}$$

with $f(n) > 0, n < p$ and $f(p) = 0$. The simplest choice is the step function mentioned after equation (5.1) or $f(N) = 1 - (N/p)$. The corresponding statistics is Bose statistics restricted to N-particle states with $N \leqslant p$. Hence, the counting rule is simply

$$D^{\mathrm{B}}(M, N) = \binom{M + N - 1}{N}$$

$N \leqslant p$ and $D^{\mathrm{B}}(M, N) = 0$ if $N > p$. Therefore, the above statistics is characterized by the Haldane statistics parameter $g = 0$

$$g_{n\to n+k} = \frac{d_n - d_{n+k}}{k} = \frac{M - M}{k} = 0 \tag{5.4}$$

if $n + k \leqslant p$. If $n + k = p + 1$, then $g_{n\to n+k} = M/(p - n + 1)$, $n = 1, 2, \ldots, p$, is fractional but not constant. Hence, this is not an example of the original Haldane exclusion statistics. The above statistics is also not of Karabali–Nair type, since $a_i^p \neq 0$, $a_i^{p+1} = 0$ but $N \leqslant p$. This would be equivalent only for the single-mode oscillator, $M = 1$.

Let us mention that Green's para-Fermi statistics [2] of order $p \in \mathbb{N}$ is also an example of this kind of exclusion statistics since at most p particles can occupy a given quantum state, $a_i^{p+1} = 0$. For a single oscillator $a^{p+1} = 0$, the extended statistical parameters are $g_{i\to j} = 0$ for $j \leqslant p$ and $g_{i\to p+1} = 1/(p+1-i)$.

In recent papers [16, 28], we have discussed these algebras and statistics in more detail.

6. Summary

In previous papers [16, 28], we defined the extended Haldane statistics parameters g, (see equation (2.6)) and the counting rules $D(M, N; \Gamma)$ (see equation (2.5)), for the generalized statistics formulated in the second quantized approach.

In this paper, we have proposed and further analysed three types of exclusion statistics, namely, the Calogero–Sutherland (C–S) type, the Gentile type and a combination of these two types of exclusion statistics.

We have started with the multimode oscillator algebra Γ_{ij}, equation (2.3), with positive definite Fock space. Introducing the appropriately defined step function Θ_{jk}, we have restricted the algebra Γ_{ij} in such a way that certain states in the Fock space of the algebra Γ_{ij} are forbidden, i.e. they have zero norms by construction.

The realization of the C–S type of exclusion statistics relies on the quon algebra, equation (3.3), and on the restriction between neighbour oscillators (placed on the 1D lattice), induced by the step function, equation (3.4). For this type of statistics, we have calculated extended statistics parameters g for the finite lattice, equation (3.7), and the infinite lattice, equation (3.9). We have also defined and calculated the average value of the extended statistics parameters, equation (3.8). Furthermore, we have calculated the counting rule, equation (3.10), and discussed its dependence on the parameters p and q, equations (3.11)–(3.13). In subsections 3.2–3.5, we have briefly described the possible generalization of the above procedure.

As an example of the Gentile type of exclusion statistics, we have considered the Bose-like algebra with $(a^\dagger)^{m+1} = 0$ and the step function $\Theta(m-N)$, equation (4.5). This is a restriction in the single-oscillator Fock space. We have found that the extended statistics parameters are not constant and that the counting rule differs from the Haldane–Wu formula. We have also mentioned the Karabali–Nair realization of the Gentile statistics.

Finally, we have described the combination of these two exclusions. As an example, we discussed parastatistics, equations (5.1) and (5.2). We have found that the extended statistics parameters are fractional but not constant.

None of the examples of exclusion presented here include the original Haldane proposal [5]. As we stressed before [16], it seems that the original Haldane statistics cannot be realized in the above sense, i.e. one cannot define the underlying operator algebra of creation and annihilation operators with positive Fock space and satisfy Haldane's requirements (fractional and constant g), *except* for *free* bosons and fermions. One should recall that the Haldane fractional exclusion statistics arises because the system is an *interacting* system and particles are topological excitations of a condensed matter state, rather than real particles which can exist outside the finite region of condensed matter. Hence, our analysis confirms the Haldane statement that the techniques of the second-quantized many-body theory cannot be applied to this type of exclusion statistics.

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References

[1] Gentile G 1940 *Nuovo Cimento* **17** 493
[2] Green H S 1953 *Phys. Rev.* **90** 170
Greenberg O W and Messiah A M L 1965 *Phys. Rev.* B **138** 1155
Greenberg O W and Messiah A M L 1965 *J. Math. Phys.* **6** 500
Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Tokyo: University of Tokyo Press)
[3] Prange R E and Girvin S M (ed) 1990 *The Quantum Hall Effects* (Berlin: Springer)
Stone M (ed) 1992 *Quantum Hall Effect* (Singapore: World Scientific)
Canright G S and Johnson M D 1994 *J. Phys. A: Math. Gen.* **27** 3579
[4] Wilczek F 1990 *Fractional Statistics and Anyon Superconductivity* (Singapore: World Scientific)
[5] Haldane F D M 1991 *Phys. Rev. Lett.* **67** 937
[6] Greenberg O W 1990 *Phys. Rev. Lett.* **64** 705
Greenberg O W 1991 *Phys. Rev.* D **43** 4111
Meljanac S and Perica A 1994 *Mod. Phys. Lett.* A **9** 3293
[7] Wu Y S 1994 *Phys. Rev. Lett.* **73** 922
[8] Protogenov A P and Verbus V A 1997 *Mod. Phys. Lett.* B **11** 283
Ho C L and Liao M J 1997 *Mod. Phys. Lett.* B **11** 461
Iguchi K 1997 *Phys. Rev. Lett.* **78** 3233
Iguchi K 1997 *Mod. Phys. Lett.* B **11** 765
Chaturvedi S and Srinivasan V 1997 *Phys. Rev. Lett.* **78** 4316
Isakov S, Arovas D, Myrheim J and Polychronakos A P 1996 *Phys. Lett.* A **212** 299
Polychronakos A P 1996 *Phys. Lett.* B **365** 202
Bhaduri R K, Murthy M V N and Srivastava M K 1996 *Phys. Rev. Lett.* **76** 165
Rajagopal A G 1995 *Phys. Rev. Lett.* 1995 **74** 1048
Fukui T and Kawakami N 1995 *Phys. Rev.* B **51** 5239
Sen D and Bhaduri R K 1995 *Phys. Rev. Lett.* **74** 3912
Isakov S 1994 *Mod. Phys. Lett.* B **8** 319
[9] He S, Xie X C and Zhang F C 1992 *Phys. Rev. Lett.* **68** 3460
Johnson M D and Canright G S 1994 *Phys. Rev.* B **49** 2947
Li D and Ouvry S 1994 *Nucl. Phys.* B **430 [FS]** 563
Su W P, Wu Y S and Yang J 1996 *Phys. Rev. Lett.* **77** 3423
Isakov S, Canright G S and Johnson M D 1997 *Phys. Rev.* B **55** 6727
[10] Dasnieres de Veigy A and Ouvry S 1994 *Phys. Rev. Lett.* **72** 600
Dasnieres de Veigy A and Ouvry S 1995 *Mod. Phys. Lett.* A **10** 1
Dasnieres de Veigy A and Ouvry S 1995 *Mod. Phys. Lett.* B **9** 271
Murthy M V N and Shankar R 1994 *Phys. Rev. Lett.* **72** 3629
Chen W and Ng Y J 1995 *Phys. Rev.* B **51** 14 479
Fayyazuddin A and Li D 1996 *Phys. Rev. Lett.* **76** 1707
Isakov S and Mashkevich S 1997 *Nucl. Phys.* B **504 [FS]** 701
[11] Bernard D and Wu Y S 1995 *New Developments of Integrable Systems and Long-Ranged Interaction Models (Nankai Lectures on Mathematical Physics)* ed M-L Ge and Y-S Wu (Singapore: World Scientific)
Isakov S 1994 *Int. J. Mod. Phys.* A **9** 2563
Murthy M V N and Shankar R 1994 *Phys. Rev. Lett.* **73** 3331
Ha Z N C 1994 *Phys. Rev. Lett.* **73** 1574
Dasnieres de Veigy A and Ouvry S 1995 *Phys. Rev. Lett.* **75** 352
Ujino H and Wadati M 1995 *J. Phys. Soc. Japan* **64** 4064
Ujino H and Wadati M 1996 *J. Phys. Soc. Japan* **65** 1203
Mashkevich S 1997 *Phys. Lett.* A **233** 30
[12] Ha Z N C 1995 *Nucl. Phys.* B **435 [FS]** 604
[13] Wu Y S and Yu Y 1995 *Phys. Rev. Lett.* **75** 890
[14] Nayak C and Wilczek F 1994 *Phys. Rev. Lett.* **73** 2740
Hatsugai Y, Kohmoto M, Koma T and Wu Y S 1996 *Phys. Rev.* B **54** 5358
[15] Karabali D and Nair V P 1995 *Nucl. Phys.* B **438 [FS]** 551
Chen W, Ng Y J and Van Dam H 1996 *Mod. Phys. Lett.* A **11** 795
Speliotopoulos A 1997 *J. Phys. A: Math. Gen.* **30** 6177
[16] Meljanac S and Milekovic M 1996 *Mod. Phys. Lett.* A **11** 3081
[17] Meljanac S and Milekovic M 1996 *Int. J. Mod. Phys.* A **11** 1391
Melic B and Meljanac S 1997 *Phys. Lett.* A **226** 22

Meljanac S, Stojic M and Svrtan D 1997 *Phys. Lett.* A **224** 319
[18] Jorgensen P, Schmitt L and Werner R 1995 *J. Funct. Anal.* **134** 33
Bozejko M and Speicher R 1994 *Math. Ann.* **300** 97
[19] Chou C 1992 *Mod. Phys. Lett.* A **7** 2685
Debergh N 1993 *Mod. Phys. Lett.* A **8** 765
[20] Meljanac S, Milekovic M and Pallua S 1994 *Phys. Lett.* B **328** 55
Bonatsos D and Daskaloyannis C 1993 *Phys. Lett.* A **8** 3727
[21] Jevicki A and van Tonder A 1996 *Mod. Phys. Lett.* A **11** 1397
[22] Palev T 1982 *J. Math. Phys.* **23** 1778
Palev T and Stoilova S 1994 *J. Phys. A: Math. Gen.* **27** 977
Palev T and Stoilova S 1994 *J. Phys. A: Math. Gen.* **27** 7387
Palev T and Stoilova S 1997 *J. Math. Phys.* **38** 2806
Okubo S 1994 *J. Math. Phys.* **35** 2785
[23] Zagier D 1992 *Comm. Math. Phys.* **147** 199
Meljanac S and Svrtan D 1996 *Comm. Math.* **1** 1
[24] Aref'eva I Ya and Volovich I V 1996 *Nucl. Phys.* B **462** 600
[25] Goulden I P and Jackson D M 1983 *Combinatorial Enumeration* (New York: Wiley)
[26] Polychronakos A P 1995 *Phys. Lett.* B **365** 202
[27] Meljanac S, Milekovic M and Perica A 1994 *Europhys. Lett.* **28** 79
Doresic M, Meljanac S and Milekovic M 1994 *Fizika* B **3** 57
[28] Meljanac S, Milekovic M and Stojic M 1998 *Mod. Phys. Lett.* A **13** 995